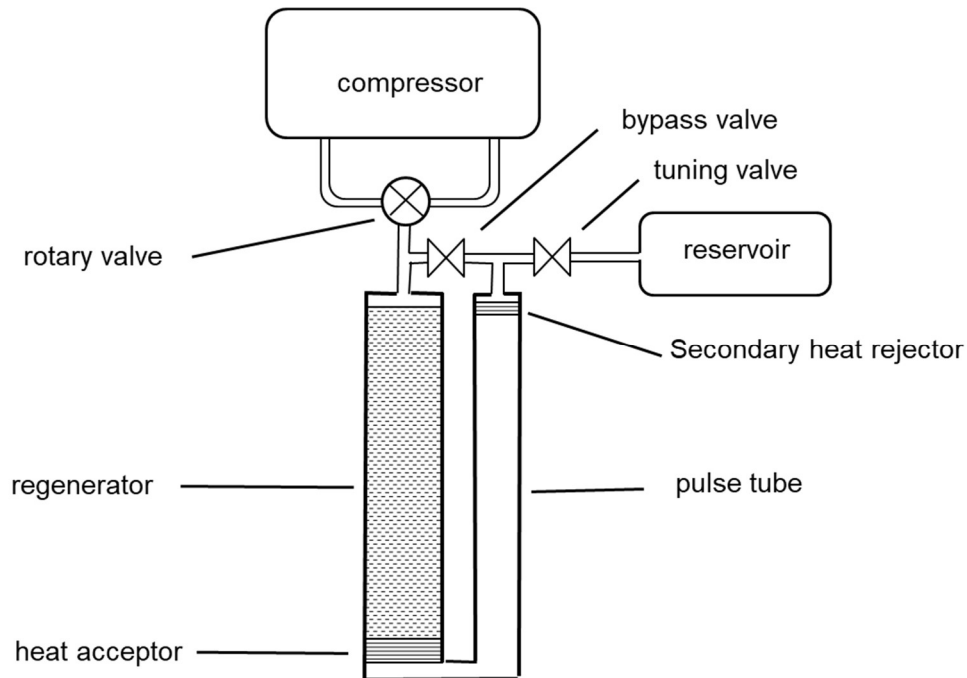


Sage Model Notes

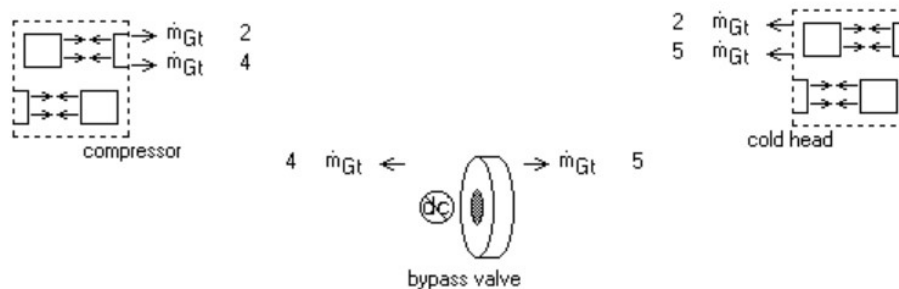
GM-PTR-SingleStage-DPreregulated.Itc

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2 September 2021

A GM pulse-tube cryocooler with a pressure regulated compressor component providing the cold-head pressure boundary condition. The model schematic looks like this:



The model is organized into submodels with the root-level model looking like this:

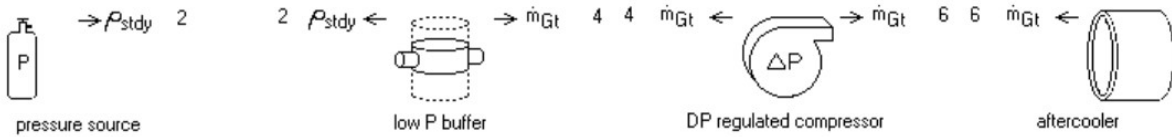


The root model includes a *bypass valve* (aka double-inlet valve) which connects the *compressor* output to the secondary rejector exit, as shown in the above schematic. The bypass valve controls the velocity amplitude within the cold head relative to the pressure amplitude — along the continuum between the velocity node of a standing wave and a traveling wave. With a properly tuned bypass valve the 4 K cooling power of this model is almost twice as much as the same model without the bypass valve. In Sage the *bypass valve* is implemented as a porous-plug orifice with the ability by numerical fiat to also control DC flow, in order to regulate any time-average flow circulation through the cold

head. The problem of controlling DC flow in an actual hardware implementation is more difficult.

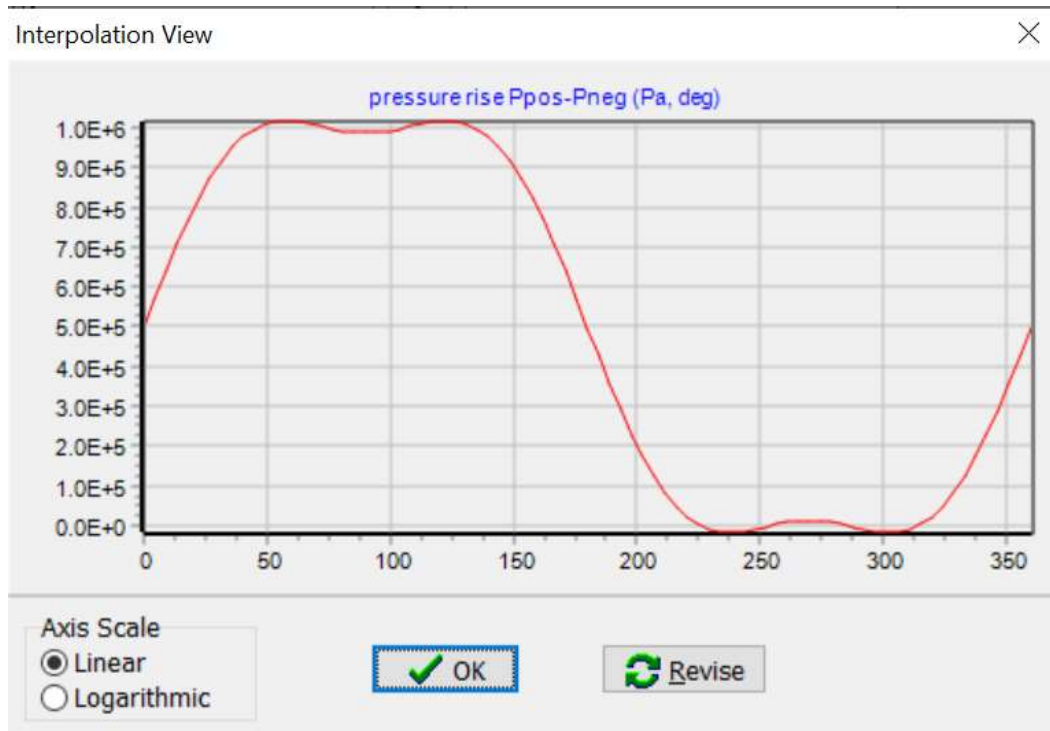
Compressor

The model components that simulate the compressor, pressure lines and rotary valve are shown in the following model view:



The *pressure-source* and *low P buffer* provide the low-side suction pressure. The *DP regulated compressor* boosts the pressure according to a pre-determined waveform. The *aftercooler* removes the adiabatic heat of compression. In the actual hardware any aftercooling is implemented upstream of the rotary valve.

The *DP regulated compressor* produces a trapezoid-shaped pressure waveform, represented by a 5-harmonic Fourier series, consistent with 11 time nodes ($NTnode = 11$, root level input). The result is visible in the interpolation view of the recast-variable dialog for input variable FDP:



The mean value, amplitudes and phases are:

$$FDP = DPmax * (HiFrac + 0.5 * TransFrac) \dots$$

(A1, A2, A3, A4, A5) Amp

(-Phase, -2*Phase, -3*Phase, -4*Phase, -5*Phase) Arg

where

```

A1          cosine coef 1          2.978E+05
  DPmax*2*(Cos(Pi*HiFrac) - Cos(Pi*(HiFrac+TransFrac)) )
/(Sqr(Pi)*TransFrac)
A2          cosine coef 2          -2.575E-14
  DPmax*2*(Cos(2*Pi*HiFrac) - Cos(2*Pi*(HiFrac+TransFrac)) )
/(Sqr(2*Pi)*TransFrac)
A3          cosine coef 3          -5.353E+04
  DPmax*2*(Cos(3*Pi*HiFrac) - Cos(3*Pi*(HiFrac+TransFrac)) )
/(Sqr(3*Pi)*TransFrac)
A4          cosine coef 4          7.724E-15
  DPmax*2*(Cos(4*Pi*HiFrac) - Cos(4*Pi*(HiFrac+TransFrac)) )
/(Sqr(4*Pi)*TransFrac)
A5          cosine coef 5          -5.493E-15
  DPmax*2*(Cos(5*Pi*HiFrac) - Cos(5*Pi*(HiFrac+TransFrac)) )
/(Sqr(5*Pi)*TransFrac)

```

The maximum pressure rise, high-pressure duration, pressure-transition duration and phase can be adjusted with these user-defined inputs:

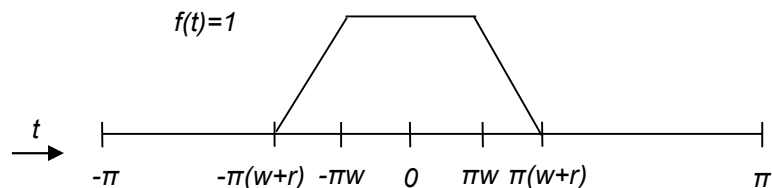
DPmax	Hi - Low pressure difference (Pa)	1.000E+06
HiFrac	fraction cycle pressure high (NonDim)	3.000E-01
TransFrac	fraction pressure transitions (NonDim)	4.000E-01
Phase	valve timing phase (deg)	9.000E+01

The sum of `HiFrac` and `TransFrac` must be less than one. $1 - \text{HiFrac} - \text{TransFrac}$ is the fraction of the cycle where there is no pressure rise. Theoretically `TransFrac` = 0 would correspond to a square-wave pressure, but `TransFrac` must be greater than zero to avoid a divide by zero error in this particular formulation.

The above amplitudes and phases approximates a linear pressure rise from zero to maximum and a linear fall from maximum to zero of the same duration. These linear transitions attempt to capture the finite opening time for the rotary valve and also the asymptotic approach to maximum and minimum pressures. More faithful representations are possible but the math would be more complicated.

Math

The waveform shape is a trapezoid and the easiest trapezoid function to analyze is defined in the interval $(-\pi, \pi)$ with the value $f(t) = 1$ for $-\pi w < t < \pi w$, rising from 0 to 1 and falling from 1 to 0 on either side of that as shown here:



w corresponds to the `HiFrac` input and r corresponds to the `TransFrac` input. The mean value is $w + r/2$ and since it is an even function the Fourier series sine coefficients b_n are all zero. The cosine coefficients are:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) dt$$

Since $f(t)$ is symmetric about zero this is just twice the integral from 0 to π

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(t) \cos(nt) dt = \frac{2}{\pi} \left[\int_0^{\pi w} \cos(nt) dt + \int_{\pi w}^{\pi(w+r)} \frac{\pi(w+r) - t}{\pi r} \cos(nt) dt \right]$$

With help from Wolfram Alpha to evaluate the integrals on the right this works out to

$$a_n = \frac{2}{n^2 \pi^2 r} [\cos(n\pi w) - \cos(n\pi(w+r))]$$

A_n is singular at $r = 0$ but approaches the limit as $r \rightarrow 0$ of

$$a_n = \frac{2}{n\pi} \sin(n\pi w)$$

Which are the cosine coefficients for a square wave.

In polar format, the amplitude coefficients are $c_n = a_n$ and phases $r_n = 0$ under the transformations:

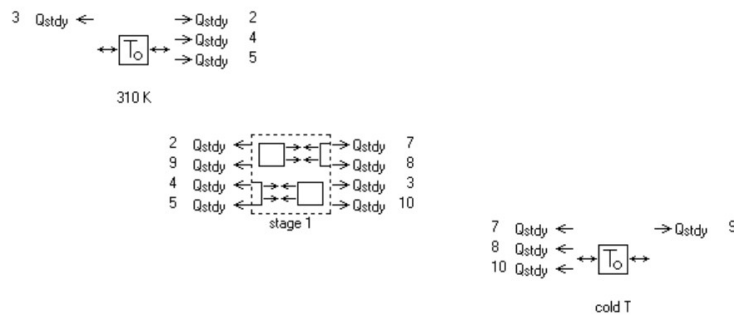
$$c_n^2 = a_n^2 + b_n^2$$

$$\tan r_n = -\frac{b_n}{a_n}$$

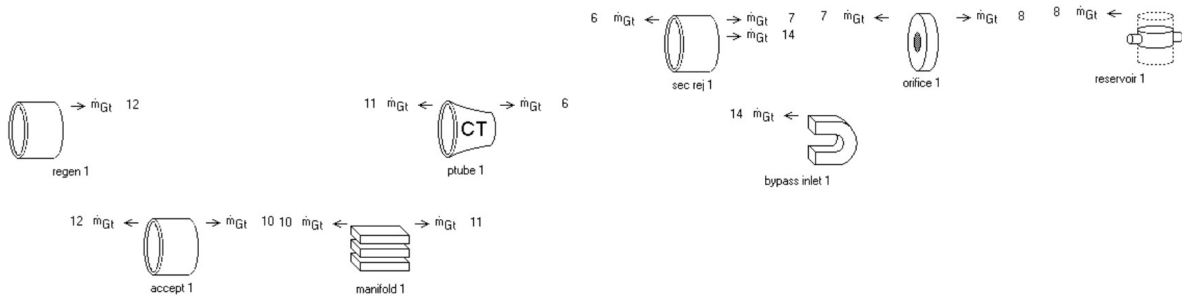
To shift the waveform phase by amount $-\theta$, r_n is shifted by amounts $-n\theta$. See *Fourier Series* in the Sage User's Guide for more information.

Cold Head

The cold-head submodel is organized into a *stage 1* submodel and two temperature sources for anchoring the components within:



Within the **stage 1** submodel are these components:



Discharge flow from the compressor enters at the negative (left) boundary of the regenerator (*regen 1*), then proceeds through the cold heat exchanger (*accept 1*), manifold (*manifold 1*), pulse tube (*ptube 1*), secondary warm heat exchanger (*sec rej 1*), orifice (*orifice 1*) and into the reservoir (*reservoir 1*). During the suction part of the cycle the flows are reversed. The orifice serves to phase-shift the flow relative to the pressure so as to accomplish an approximation of the stirling cooling cycle — roughly speaking, so the pressure variation is in phase with the velocity variation in the regenerator. The bypass inlet (*bypass inlet 1*) serves to redirect the flow from the secondary warm heat exchanger to or from the *bypass valve* at the root level. The components are arranged in rows, with colder components at the bottom. The “1” suffix in all the component names designates that this is the first stage. If this were a two-stage cold-head there might be a *stage 2* submodel with “2” suffixes for all components.

Cold Temperature The file is currently set up for a cold-end temperature of 50K. To change it change the *Tinit* input for the *cold T* heat source, which anchors the temperature of *accept 1* and manifold 1 as well as the cold ends of the thermal conduction walls in *regen 1* and *ptube 1*. Also, you might want to change *Tinit* at the positive end of *regen 1* and negative end of the *ptube 1*. *Tinit* changes are not strictly necessary because the regenerator and pulse-tube temperatures are solved, with *Tinit* only providing initial values.

Net cooling power is available in the *cold head* as user-defined variable *Qlift*. Included in *Qlift* are the heat flows absorbed by the helium in the cold hx, less the conduction losses down the regenerator and pulse-tube canisters.

Optimization

This model implements a simple optimization to maximize the cooling power subject to 1 kW compressor power. The cooling power is defined in the *cold head* submodel as

```
Qlift          net heat lift          2.992E+01
QcoldT
```

The compressor power is defined in the *DP regulated compressor* component of the compressor submodel as

```
wcompressor   adiabatic compressor power  -1.000E+03
FWc.Mean
```

FWc is a built-in Fourier series output variable.

The optimized variables are the bypass valve length, the cold head length (L_{stage1}) regenerator and pulse-tube diameters (D_{reg} and D_{ptb}) and the length of the tuning orifice ($Length$).